

General Theorem about Gravitational Lensing

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ABSTRACT

We study the general theorem about gravitational lensing which states the relationship between the numbers of images with different parities. Our formulation allows an extension to the nontransparent and singular model.

Subject headings: gravitational lensing

1. Introduction

There are the general theorems about the gravitational lensing (MacKenzie 1985 ; Schneider, Ehlers, & Falco 1992) which mainly state the relationships between the numbers of the images with different parities;

$$n_I - n_{II} + n_{III} = 1, \quad (1)$$

where n_I is the number of images of type I. We use n_{II} and n_{III} analogously.

The types of images are defined by the Jacobian matrix of the Fermat's potential ϕ ,

$$A_{ij} \equiv \frac{\partial^2 \phi}{\partial x_i \partial x_j} \quad (i, j = 1, 2). \quad (2)$$

Type I : both eigenvalues of the Jacobian matrix are positive.

Type II : one eigenvalue is positive and another negative.

Type III : both are negative.

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Therefore type I and type III images have positive parities and type II images have negative ones.

The existing general theorems are general for transparent and non-singular lens models. Typical examples are the elliptical lens models with softened power law behavior (Binney & Tremaine 1987 ; Schneider, et al. 1992). However, they do not exhaust all the lens models. They are rather phenomenological models, whereas there is a widely valid model. It is multipole expansion model (MPE model) which is obtained by expanding deflection potential with respect to impact parameter (Yoshida & Omote 1988 ; Kakigi, Okamura, & Fukuyama 1995 ; Fukuyama, Kakigi, & Okamura 1997). The latter is less popular than the former in lensing analysis but is very powerful (Fukuyama, et al. 1997). The MPE model is nontransparent and singular, and we see how the general theorems are extended to incorporate MPE model and other models. In this extension we propose the generalized index theorem valid also to the nontransparent and singular lens model and the formulation in complex coordinates is found to be very convenient.

2. Formulation

Before proceeding to the concrete physical situation, let us begin with the mathematical preparations. Index theorem for critical points in two dimensional flat or closed curved surfaces is well known for mathematicians (Eguchi, Gilkey, & Hanson 1980). They are summarized as follows: We consider the vector field on the surface S . Denoting the critical points (at which the vector field vanishes) by P_i on S , we get the following theorem.

$$\sum_i (\text{Index of the vector at } P_i) = \chi(S), \quad (3)$$

where $\chi(S)$ is the Euler characteristic of S . For flat space χ is one and the result becomes equation (1). This theorem does not hold in the case that the vector field has singular points. So we reformulate the index theorem so as to incorporate pole terms.

Let us consider a complex function $f(z)$ with a complex variable z that is not necessarily holomorphic. The function f plays the role of the vector field mentioned above. The index of f is defined by

$$\text{Ind}(f) \equiv \frac{1}{2\pi i} \oint_{C_z} d \ln (f(z)) . \quad (4)$$

This measures the variation of the phase of f along the closed curve C_z which goes anti-clockwise. When C_z does not enclose any zero points or poles, the index vanishes. If C_z encloses some zero points or poles, $\{P_i\}$, then the index is given by the summation of the contributions from each point P_i ,

$$\text{Ind}(f) = \sum_i \text{Ind}(f ; P_i), \quad (5)$$

$$\text{Ind}(f ; P_i) = \frac{1}{2\pi i} \oint_{C_z^{(i)}} d \ln (f(z)) , \quad (6)$$

where $C_z^{(i)}$ is the circle enclosing the point P_i . In the context of the gravitational lenses, f is replaced by $\phi_{\bar{z}}(\equiv \partial\phi/\partial\bar{z}$; \bar{z} is complex conjugate of z) and $\phi_{\bar{z}} = 0$ is just lens equation (see eqs.[14] and [15]). Therefore ordinary images are zero points of order one. If lens object has finite total mass and extended regular mass density, then we have the pole only at infinity. Whereas in the case of MPE models and models with singular mass density such as point masses, other poles are produced.

From the form of equation (5) we should pay attention to each contribution. Let us consider the map $\omega = f(z)$ in the neighborhood of P_i . Then the $\text{Ind}(f ; P_i)$ is rewritten as

$$\text{Ind}(f ; P_i) = \frac{1}{2\pi i} \oint_{C_\omega} d \ln \omega, \quad (7)$$

where $C_\omega = \omega(C_z^{(i)})$. The value of the right hand side is the (algebraic) number of times that we wind around C_ω when we go around $C_z^{(i)}$ in the anti-clockwise direction.

Firstly we consider zero point case and proceed to the case of poles. Hereafter we set the origin of complex z -plane at the zero point P_i and we use bold-face letters for the value of any function g , for example, $\mathbf{g} = g(P_i)$.

2.1. The case of zero point of order one

In the neighborhood of a zero point, ω can be written as

$$\omega = z\mathbf{f}_z + \bar{z}\mathbf{f}_{\bar{z}} + O(|z|^2), \quad (8)$$

where lower indices z and \bar{z} mean partial derivatives with respect to them, for example,

$$f_{z^k \bar{z}^l} \equiv \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} f.$$

We define μ by

$$\mu \equiv \frac{\mathbf{f}_{\bar{z}}}{\mathbf{f}_z}.$$

Then the Jacobian D of the mapping from z to ω is in this case given by

$$D = |\mathbf{f}_z|^2 - |\mathbf{f}_{\bar{z}}|^2 = |\mathbf{f}_z|^2(1 - |\mu|^2). \quad (9)$$

That is, $\omega = f(z)$ reserve (reverse) the direction of rotation if $|\mu| < 1$ ($|\mu| > 1$).

Winding number of C_ω is one for the point of order one and

$$\text{Ind}(f ; P_i) = \begin{cases} 1 & \text{for } |\mu| < 1 \\ -1 & \text{for } |\mu| > 1. \end{cases} \quad (10)$$

2.2. The case of zero point of higher orders

About a zero point, ω is expanded as a polynomial of z and \bar{z} . However the winding number is determined by the lowest power term as is easily checked.

For the simplicity, we consider the case that dominant part in the neighborhood of the zero point consists of the only term as

$$\omega = \frac{\mathbf{f}_{z^k \bar{z}^l}}{k! l!} z^k \bar{z}^l + (\text{subdominant terms}) \quad (11)$$

We call this type of zero point as (k, l) -type critical point for later use. Although equation (11) restricts the form of f , as we see later, for the case of point masses and MPE models, it is enough to examine equation (11).

Then the index of f is given by

$$\text{Ind}(f ; P_i) = k - l. \quad (12)$$

Therefore, from the previous subsection 2.1, we can regard zero points with $|\mu| < 1$ as $(1, 0)$ -type and ones with $|\mu| > 1$ as $(0, 1)$ -type.

2.3. The case of pole terms

As was mentioned, besides the zero points, pole terms of f also contribute to $\text{Ind}(f)$. In the neighborhood of pole of f we define g by $g \equiv 1/f$, then

$$\text{Ind}(f ; P_i) = -\text{Ind}(g ; P_i) \quad (13)$$

comes from the definition of equation (4). If f consists of several pole terms, $\text{Ind}(f)$ is determined by the highest pole term.

Thus, similarly to the case of zero points, we can regard the pole whose dominant term has the form of equation (11) as (k, l) -type critical point with negative integers k and l .

3. Application to gravitational lensing

We have completed mathematical preparations and proceed to application to gravitational lensing. In this case f is defined through the Fermat's potential ϕ by

$$f = \phi_{\bar{z}}, \quad (14)$$

where

$$\phi = \frac{1}{2}|z - z_s|^2 - \psi(z). \quad (15)$$

$z(z_s)$ are image (source) positions in the complex notations and ψ is the deflection potential defined by

$$\psi(z) \equiv \frac{1}{\pi} \int d^2 z' \kappa(z') \ln |z - z'|, \quad (16)$$

where the integral measure $d^2 z$ means $d\Re(z)d\Im(z)$ and κ is normalized surface mass density of lens object. We assume

$$\lim_{|z| \rightarrow \infty} \left| \frac{\psi_{\bar{z}}}{z} \right| = 0, \quad (17)$$

and this condition may hold for the models with finite total mass.

For the large impact parameter, $\phi_{\bar{z}}$ becomes

$$\phi_{\bar{z}} = \frac{1}{2}(z - z_s) + o(|z|). \quad (18)$$

That is, infinite point becomes pole. Remarking that the anti-clockwise loop C_z ($|z| \gg 1$) is viewed as clockwise one from the infinite point, we obtain the contribution of the infinite point to the index as

$$\text{Ind}(\phi_{\bar{z}}) = 1. \quad (19)$$

Diminishing the radius of the circle C_z , it is entangled with the zero points and poles. Then equation (19) must be equal to the sum of these contributions.

As we have mentioned, the zero point of order one in the neighborhood of the origin is classified into $(1, 0)$ -type for $|\mu| < 1$ and $(0, 1)$ -type for $|\mu| > 1$ by the contribution to the index. Thus, we can say that the contribution comes from the term

$$\omega \sim \frac{C_{k,l}}{k! l!} z^k \bar{z}^l,$$

where $C_{k,l}$ is a constant and positive (negative) k and l correspond to the zero points (poles). We denote the number of (k, l) -type critical points as $n^{(k,l)}$. Then from the arguments of the section 2, we get

$$1 = \text{Ind}(\phi_{\bar{z}}) = \sum_{k,l} (k - l) n^{(k,l)}. \quad (20)$$

- For the extended lens models which have no pole we obtain

$$1 = n^{(1,0)} - n^{(0,1)} = n_+ - n_- = n_I + n_{III} - n_{II}, \quad (21)$$

where $n_+(n_-)$ is the number of the images with the positive(negative) parity.

- For the point masses, we have a pole of $(0, -1)$ -type for every point mass and therefore

$$1 = n_I + n_{III} - n_{II} + (\# \text{ of masses}). \quad (22)$$

- For a MPE model up to 2^k -pole, Fermat's potential is given by (Kakigi, et al. 1995)

$$\phi = \frac{1}{2}|z - z_s|^2 - \frac{1}{2}\ln(z\bar{z}) + \sum_{n=1}^k \left(\frac{Q_n}{z^n} + \frac{\bar{Q}_n}{\bar{z}^n} \right) \quad (23)$$

and the vector field becomes

$$\phi_{\bar{z}} = \frac{1}{2}(z - z_s) - \frac{1}{2\bar{z}} - \frac{1}{2} \sum_{n=1}^k \left(n \frac{\bar{Q}_n}{\bar{z}^{n+1}} \right). \quad (24)$$

So there give rise a pole of $(0, -k - 1)$ -type and

$$1 = n^{(1,0)} - n^{(0,1)} + (k + 1) n^{(0,-k-1)} = n_I + n_{III} - n_{II} + k + 1. \quad (25)$$

Lastly, we give the concrete example of the theorem in the multiple images of PG1115+080 (Young, et al. 1981 ; Christian, Crabtree, & Waddell 1987 ; Kristian, et al. 1993 ; Schechter, et al. 1997). Fig. 1 shows critical and caustics lines of the Fermat's potential and image positions numerically fitted by a model with softened power law surface density (Asano & Fukuyama 1996). Critical lines are those on which Jacobian determinant equation (2) vanishes. So the parities of images are reversed across a critical line. Images outside the outermost critical line have positive parities. Hence critical lines are useful for determination of parities of images.

From positions of the critical lines, we can know that A_1 and C belong to n_I , A_2 and B do to n_{II} and D to n_{III} . So the calculated numbers of images satisfy the original theorem equation (1).

On the other hand Fig. 2 shows fitting of the same lens system by a MPE model including up to quadrupole (Fukuyama, et al. 1997). A_2 and B belong to n_I . The other A_1 , C , D and E do to n_{II} . Thus, these satisfy equation (25) for the case of quadrupole, $k = 2$.

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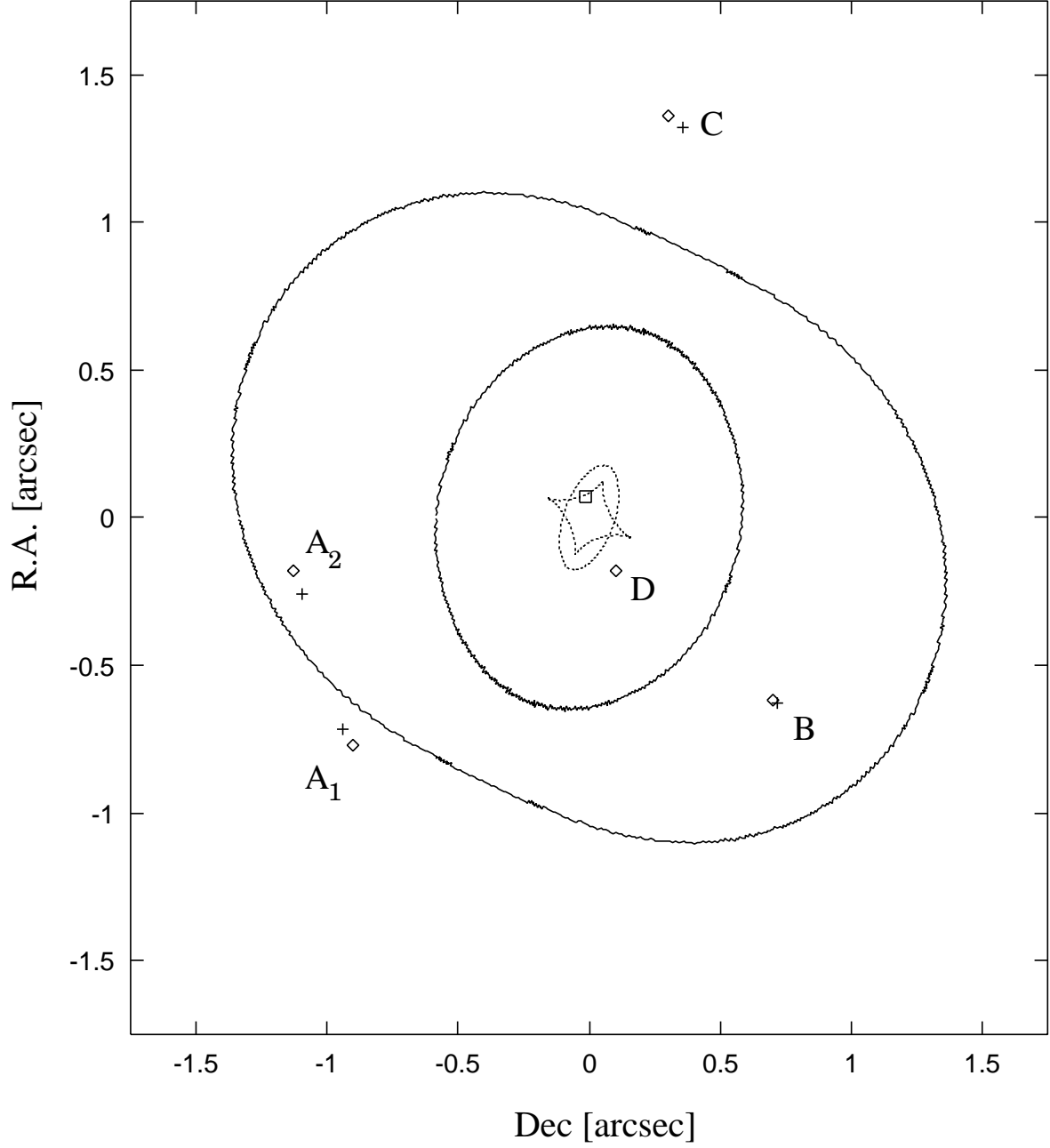


Fig. 1.— The observed image positions (crosses) and numerically fitted ones(diamonds) by a model with softened power law surface mass density. The square shows source position. The solid and dotted lines are critical line and caustics, respectively.

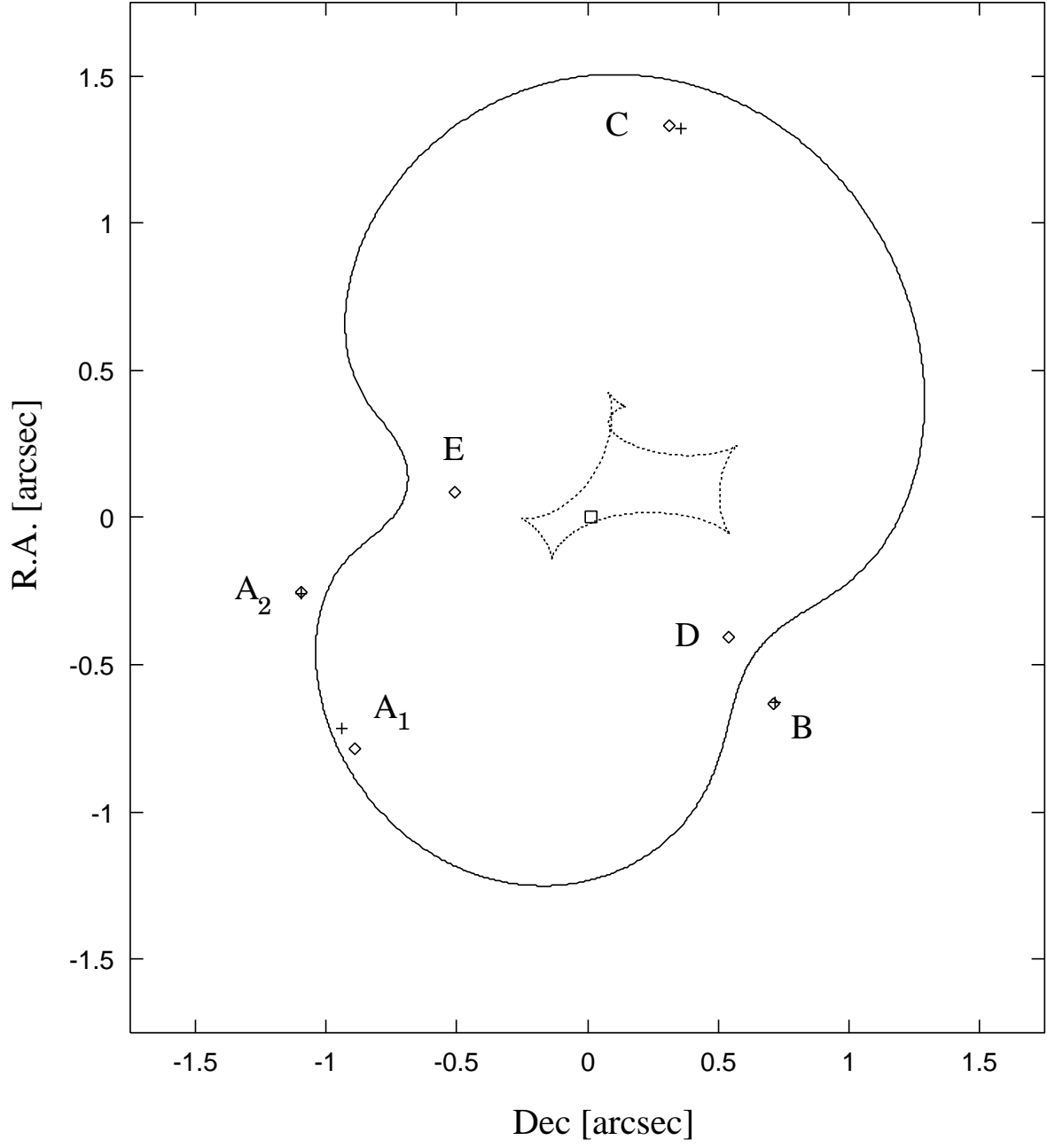


Fig. 2.— ibid. by a MPE model.